BNL-107540-2015-IR

C-A/AP/536
February 2015

Ion Emittance Growth Due to Focusing Modulation from Slipping Electron Bunch

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Low energy RHIC operation has to be operated at an energy ranging from $\gamma = 4.1$ to $\gamma = 10$. The energy variation causes the change of revolution frequency. While the rf system for the circulating ion will operate at an exact harmonic of the revolution frequency ($h = 60$ for 4.5 MHz rf and $h = 360$ for 28 MHz rf.), the superconducting rf system for the cooling electron beam does not have a frequency tuning range that is wide enough to cover the required changes of revolution frequency. As a result, electron bunches will sit at different locations along the ion bunch from turn to turn, i.e. the slipping of the electron bunch with respect to the circulating ion bunch. At cooling section, ions see a coherent focusing force due to the electrons' space charge, which differs from turn to turn due to the slipping. We will try to estimate how this irregular focusing affects the transverse emittance of the ion bunch.

Time Domain Analysis

Assuming the line charge distribution of the electrons and the ions are Gaussian, their currents can be written as

$$I_e(t) = \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\left(t - n\cdot\Delta T_e\right)^2}{2\sigma_e^2}\right], \tag{1}$$

and[1]

$$I_i(t) = \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{m=-\infty}^{\infty} \exp\left[-\frac{\left(t - m\cdot T_{rev}\right)^2}{2\sigma_e^2}\right]. \tag{2}$$

The electron current as observed by an ion bunch at its Mth revolution is given by

$$\begin{aligned} I_e(M,\tau) &= \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\left(M\cdot T_{rev} + \tau - n\cdot\Delta T_e\right)^2}{2\sigma_e^2}\right] \\ &= \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{\tilde{n}=-\infty}^{\infty} \exp\left\{-\frac{\left[\tau - \tilde{n}\cdot\Delta T_e + \left(M\cdot h_e - Round\left(M\cdot h_e\right)\right)\Delta T_e\right]^2}{2\sigma_e^2}\right\}, \end{aligned} \tag{3}$$

where

$$h_e = \frac{f_{e,rep}}{f_{rev}} \tag{4}$$

with $f_{e,rep}$ being the repetition frequency of the electron beam and f_{rev} is RHIC revolution frequency, i.e. ~78 KHz. The ions sitting at the center of the bunch, i.e. $\tau = 0$, get a kick at its Mth revolution of

[1] We consider one circulating ion bunch.

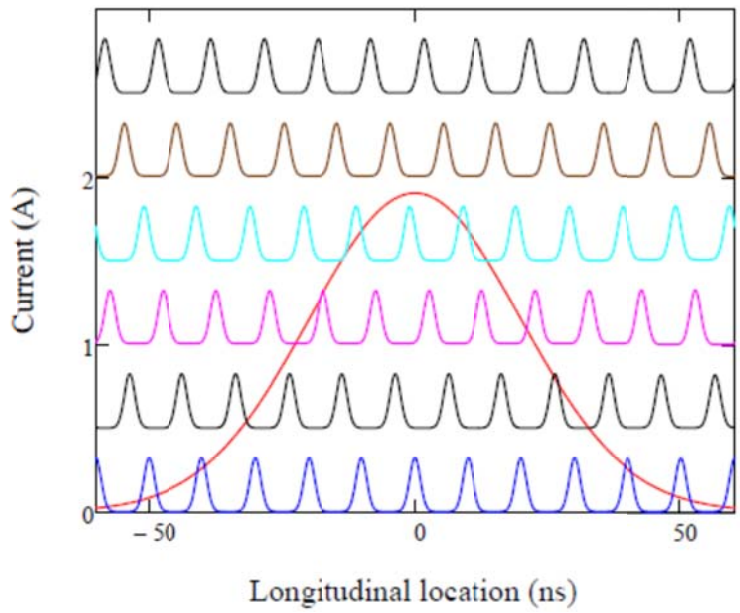

Figure 1: mountain range plot of electron current evolution for 5 successive turns as observed by an ion bunch at the cooling section. The electron current is calculated from eq. (3) for $\gamma_i = 4.1$, $\sigma_e = 1ns$, $\Delta T_e = 10ns$, $Q_e = 0.8nC$, $f_{i,rf} = 4.56MHz$, $\sigma_i = 5.8m/(\beta_i c) = 19.95ns$ and $N_i = 0.75\times10^9$.

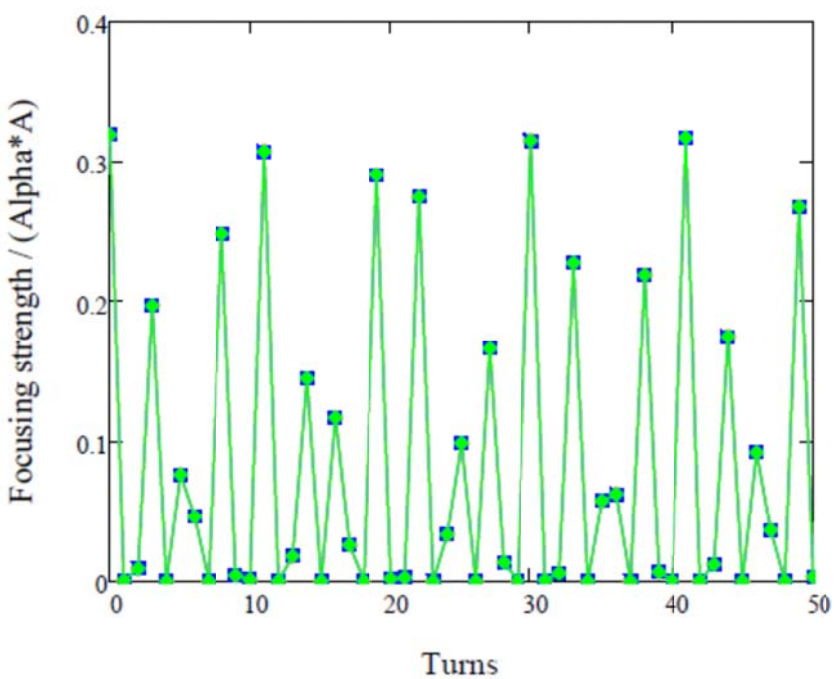

Figure 2: focusing strength seen by ions at the center of the ion bunch in the cooling section. The abscissa is the independent time variable in number of turns and the ordinate is the focusing strength of the electrons at the location of the ion bunch center as calculated from eq. (5) and (8) (they overlap).

$$\varepsilon_M = \alpha \cdot I_e(M,0)$$
$$= \alpha \cdot \frac{Q_e}{\sqrt{2\pi}\sigma_e} \sum_{\tilde{n}=-\infty}^{\infty} \exp\left\{ -\frac{\left[-\tilde{n}\cdot \Delta T_e + \left(M \cdot h_e - Round\left(M \cdot h_e\right)\right)\Delta T_e\right]^2}{2\sigma_e^2} \right\}. \quad (5)$$

Assuming $\Delta T_e >> \sigma_e$, only the closest electron bunch, with $\tilde{n} = 0$, has contribution and hence eq. (5) simplifies to

$$\varepsilon_M = \alpha \cdot \frac{Q_e}{\sqrt{2\pi}\sigma_e} \exp\left\{ -\frac{\left[M \cdot h_e - Round\left(M \cdot h_e\right)\right]^2 \Delta T_e^2}{2\sigma_e^2} \right\}. \quad (6)$$

Splitting h_e into the integer part, $h_{e,I}$, and the fractional part, ν_e , i.e.

$$h_e = h_{e,I} + \nu_e, \quad (7)$$

eq. (6) becomes

$$\varepsilon_M = \alpha \cdot \frac{Q_e}{\sqrt{2\pi}\sigma_e} \exp\left\{ -\frac{\left[M \cdot \nu_e - Round\left(M \cdot \nu_e\right)\right]^2 \Delta T_e^2}{2\sigma_e^2} \right\}. \quad (8)$$

The coefficient α can be derived as (See APPENDIX),

$$\alpha = \frac{Z_i e^2 L_{cool}}{2\varepsilon_0 m_i \gamma^3 \omega_x \pi a_e^2 (\beta c)^2} = \frac{2 r_0 \beta_{avg}}{\gamma^3 \beta^2 a_e^2} \frac{L_{cool}}{\beta c e}, \quad (9)$$

where

$$\beta_{avg} = \frac{\beta c}{\omega_x}, \quad (10)$$

and

$$r_0 = \frac{Z_i e^2}{4\pi\varepsilon_0 m_i c^2}. \quad (11)$$

Inserting eq. (9) into eq. (8) leads to

$$\varepsilon_M = \frac{2 r_0 \beta_{avg}}{\gamma^3 \beta^2 a_e^2} \frac{L_{cool}}{\beta c e} \cdot \frac{Q_e}{\sqrt{2\pi}\sigma_e} \exp\left\{ -\frac{\left[M \cdot \nu_e - Round\left(M \cdot \nu_e\right)\right]^2 \Delta T_e^2}{2\sigma_e^2} \right\}. \quad (12)$$

Applying eq. (77) to eq. (12), we obtain

$$\varepsilon_M = 4\pi \cdot \Delta\nu_{e,M}$$
$$= 4\pi \cdot \Delta\nu_{peak} \cdot \exp\left\{ -\frac{\left[M \cdot \nu_e - Round\left(M \cdot \nu_e\right)\right]^2 \Delta T_e^2}{2\sigma_e^2} \right\} \quad (13)$$

with ν_e is the fractional part of $f_{e,rep} / f_{rev}$ and

$$\Delta\nu_{peak} = \alpha \cdot \frac{Q_e}{4\pi\sqrt{2\pi}\sigma_e} = \frac{N_e}{2\pi\sqrt{2\pi}\sigma_e} \frac{r_0 \beta_{avg}}{\gamma^3 \beta^3 a_e^2} \frac{L_{cool}}{c}. \quad (14)$$

Using eq. (9) from [1], we have

$$\frac{J\left(N,\hat{\psi}\right)}{J\left(0\right)}=\exp\left[u\left(N,\hat{\psi}\right)\right], \tag{15}$$

with

$$u\left(N,\hat{\psi}\right)=-\sum_{k=0}^{N}\sin\left(4\pi k\tilde{Q}+2\hat{\psi}\right)\delta\varepsilon_k+\frac{1}{2}\left[\sum_{k=0}^{N}\cos\left(4\pi k\tilde{Q}+2\hat{\psi}\right)\delta\varepsilon_k\right]^2, \tag{16}$$

$$\varepsilon_{avg}=\frac{1}{N}\sum_{k=1}^{N}\varepsilon_k\ , \tag{17}$$

$$\tilde{Q}=Q+\frac{\varepsilon_{avg}}{4\pi}\ , \tag{18}$$

and

$$\delta\varepsilon_k=\varepsilon_k-\varepsilon_{avg}\ . \tag{19}$$

The evolution of the phase is given by

$$\begin{aligned}\psi_N&=2\pi\tilde{Q}N+\hat{\psi}+\sum_{i=0}^{N}\sin^2\psi_i\delta\varepsilon_i\\&\approx 2\pi\tilde{Q}N+\hat{\psi}-\frac{1}{2}\sum_{i=0}^{N}\cos\left[4\pi\tilde{Q}i+2\hat{\psi}\right]\delta\varepsilon_i\end{aligned} \tag{20}$$

Fourier Analysis

In order to find out the resonant condition for emittance growth, we Fourier decompose eq. (1) as

$$\begin{aligned}I_e\left(t\right)&=\frac{Q_e}{\sqrt{2\pi}\sigma_e}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{\left(t-n\cdot\Delta T_e\right)^2}{2\sigma_e^2}\right]\\&=\frac{Q_e}{\sqrt{2\pi}\sigma_e}\left\{\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2k\pi t}{\Delta T_e}\right)+b_k\sin\left(\frac{2k\pi t}{\Delta T_e}\right)\right]\right\}\end{aligned}, \tag{21}$$

with

$$a_k\approx\frac{2\sqrt{2\pi}\sigma_e}{\Delta T_e}\exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right), \tag{22}$$

and

$$b_k\approx 0\,. \tag{23}$$

for $\sigma_e << \Delta T_e$. Inserting eq. (22) and (23) into eq. (21) leads to

$$\begin{aligned}I_e\left(t\right)&=\frac{Q_e}{\sqrt{2\pi}\sigma_e}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{\left(t-n\cdot\Delta T_e\right)^2}{2\sigma_e^2}\right]\\&=\frac{2Q_e}{\Delta T_e}\left\{\frac{1}{2}+\sum_{k=1}^{\infty}\exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\cos\left(\frac{2k\pi t}{\Delta T_e}\right)\right\}\end{aligned}. \tag{24}$$

The average electron current seen by ions is thus

$$\langle I_e \rangle_{avg} = \frac{2Q_e}{\Delta T_e}\left\{\frac{1}{2}+\sum_{k=1}^{\infty}\Theta(N,\nu_e)\exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\right\}, \tag{25}$$

with

$$\Theta(N,\nu_e) \equiv \frac{1}{2(N+1)}\left\{1+\frac{\sin\left[(2N+1)k\pi\nu_e\right]}{\sin(k\pi\nu_e)}\right\}. \tag{26}$$

Inserting eq. (24) and (25) into eq. (5) and (19) yields

$$\delta\varepsilon_l = \frac{2\alpha Q_e}{\Delta T_e}\sum_{k=1}^{\infty}\exp\left(-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}\right)\cdot\left[\cos\left(\frac{2k\pi l T_{rev}}{\Delta T_e}\right)-\Theta(N,\nu_e)\right]. \tag{27}$$

Applying eq. (27) to the first term of eq. (16) produces

$$\begin{aligned} u_1(N,\hat{\psi}) &= -\frac{4(2\pi)^{\frac{3}{2}}\sigma_e\Delta\nu_{peak}}{\Delta T_e}\sum_{l=0}^{N}\sin\left(4\pi l\tilde{Q}+2\hat{\psi}\right)\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\cdot\left[\cos\left(\frac{2k\pi l T_{rev}}{\Delta T_e}\right)-\Theta(N,\nu_e)\right] \\ &= -\frac{2(2\pi)^{\frac{3}{2}}\sigma_e\Delta\nu_{peak}}{\Delta T_e}\left\{-2\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\Theta(N,\nu_e)\sum_{l=0}^{N}\sin\left(4\pi l\tilde{Q}+2\hat{\psi}\right)+\right. \\ &\left.\sum_{l=0}^{N}\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\sin\left[2\pi l\left(2\tilde{Q}+\frac{kT_{rev}}{\Delta T_e}\right)+2\hat{\psi}\right]+\sum_{l=0}^{N}\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\sin\left[2\pi l\left(2\tilde{Q}-\frac{kT_{rev}}{\Delta T_e}\right)+2\hat{\psi}\right]\right\} \end{aligned} \tag{28}$$

The second term of eq. (16) reads

$$\begin{aligned} u_2(N,\hat{\psi}) &= \frac{1}{2}\left[\sum_{m=0}^{N}\cos\left(4\pi m\tilde{Q}+2\hat{\psi}\right)\delta\varepsilon_m\right]^2 \\ &= 64\pi^3\left(\frac{\sigma_e\Delta\nu_{peak}}{\Delta T_e}\right)^2\sum_{l,k=1}^{\infty}\mathrm{e}^{-\frac{2\left(k^2+l'^2\right)\pi^2\sigma_e^2}{\Delta T_e^2}}\sum_{m,n=0}^{N}\left[\cos\left(\frac{2k\pi m T_{rev}}{\Delta T_e}\right)-\Theta(N,k\nu_e)\right]. \\ &\left[\cos\left(\frac{2l\pi n T_{rev}}{\Delta T_e}\right)-\Theta(N,l\nu_e)\right]\cos\left(4\pi n\tilde{Q}+2\hat{\psi}\right)\cos\left(4\pi m\tilde{Q}+2\hat{\psi}\right) \end{aligned} \tag{29}$$

For $\nu_e \neq 0$, taking the leading term of $1/N$ in eq. (29) leads to

$$\begin{aligned} u_2(N,\hat{\psi}) &\approx 8\pi^3\left(\frac{\sigma_e\Delta\nu_{peak}}{\Delta T_e}\right)^2\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{4k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\sum_{m,n=0}^{N} \\ &\left\{\cos\left[2\pi(n+m)\cdot\left(2\tilde{Q}+\frac{T_{rev}}{\Delta T_e}k\right)+4\hat{\psi}\right]+\cos\left[2\pi(n+m)\cdot\left(2\tilde{Q}-\frac{T_{rev}}{\Delta T_e}k\right)+4\hat{\psi}\right]\right. \\ &\left.+\cos\left[2\pi(n-m)\cdot\left(2\tilde{Q}+\frac{T_{rev}}{\Delta T_e}k\right)\right]+\cos\left[2\pi(n-m)\cdot\left(2\tilde{Q}-\frac{T_{rev}}{\Delta T_e}k\right)\right]\right\} \end{aligned} \tag{30}$$

Hence the resonant condition is

$$2\tilde{Q} \pm \frac{kT_{rev}}{\Delta T_e} = K\,, \tag{31}$$

with K being an integer. Letting $2Q_{dec}$ be the fractional part of $2\tilde{Q}$ and ν_e be the fractional part of $\frac{f_{rep}}{f_{rev}}$, eq. (31) becomes

$$2Q_{dec} = K \pm k\nu_e\,. \tag{32}$$

If $2\tilde{Q} - \frac{T_{rev}}{\Delta T_e}k = 0$, eq. (28) becomes

$$u_1\left(N,\hat{\psi}\right) = -\frac{2\left(2\pi\right)^{\frac{3}{2}}\sigma_e \Delta\nu_{peak}}{\Delta T_e} N\,\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}} \sin\left(2\hat{\psi}\right) \tag{33}$$

eq. (29) becomes

$$u_2\left(N,\hat{\psi}\right) \approx 8\pi^3\left(\frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e}\right)^2 N^2\,\mathrm{e}^{-\frac{4k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\left[1+\cos\left(4\hat{\psi}\right)\right]. \tag{34}$$

Inserting eq. (33) and (34) into eq. (15) yields

$$\frac{J\left(N,\hat{\psi}\right)}{J\left(0\right)} = \exp\left[u_k\left(N,\hat{\psi}\right)\right], \tag{35}$$

where

$$u_k\left(N,\hat{\psi}\right) = -2g_k N\sin\left(2\hat{\psi}\right) + g_k{}^2 N^2\left[1+\cos\left(4\hat{\psi}\right)\right], \tag{36}$$

and

$$g_k \equiv \left(2\pi\right)^{\frac{3}{2}}\frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e}\mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}}\,. \tag{37}$$

Inserting eq. (36) into eq. (35) and averaging over the initial phase yields

$$\begin{aligned} J_{avg}\left(N\right) &= J\left(0\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left[u_k\left(N,\hat{\psi}\right)\right]d\hat{\psi} \\ &= J\left(0\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{-2g_k N\sin\left(2\hat{\psi}\right) + g_k{}^2 N^2\left[1+\cos\left(4\hat{\psi}\right)\right]\right\}d\hat{\psi} \end{aligned}\,. \tag{38}$$

For $u_k\left(N,\hat{\psi}\right) << 1$, eq. (38) becomes

$$\begin{aligned} J_{avg}\left(N\right) &= J\left(0\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{-2g_k N\sin\left(2\hat{\psi}\right) + g_k{}^2 N^2\left[1+\cos\left(4\hat{\psi}\right)\right]\right\}d\hat{\psi} \\ &\approx \frac{J\left(0\right)}{2\pi}\int_0^{2\pi}\left\{1 + g_k{}^2 N^2 + 2g_k{}^2 N^2\sin^2\left(2\hat{\psi}\right)\right\}d\hat{\psi} \\ &= J\left(0\right)\left(1+2g_k{}^2 N^2\right) \end{aligned}\,. \tag{39}$$

Simple Tracking

Using the normalized canonical variables:

$$P_x = \frac{p_x}{\sqrt{Q_x}} = \sqrt{2J}\cos\psi \tag{40}$$

and

$$X = x\sqrt{Q_x} = \sqrt{2J}\sin\psi \ , \tag{41}$$

the one turn transfer map can be written as

$$X_n(k+1) = \cos(2\pi Q_x) X_n(k) + \sin(2\pi Q_x) P_n(k) \tag{42}$$

and

$$P_n(k+1) = \cos(2\pi Q_x) P_n(k) - \sin(2\pi Q_x) X_n(k) + \varepsilon_k X_n(k+1). \tag{43}$$

The tracking is done in Mathcad and the results are shown in fig. 7 for the relative growth of the average action growth of 300 macro-particles with the same initial action and evenly uniformly distributed phases. The betatron tune is taken as $Q_x = 28.15$ and the electron bunch repetition frequency is taken such that the fractional part of $f_{e,rep} / f_{rev}$ is 0.3. As shown in fig. 3, the square growth formula, eq. (39) fully agrees with the tracking results. Fig. 8 shows the phase space plot comparison between the initial and final distribution after 2000 turns.

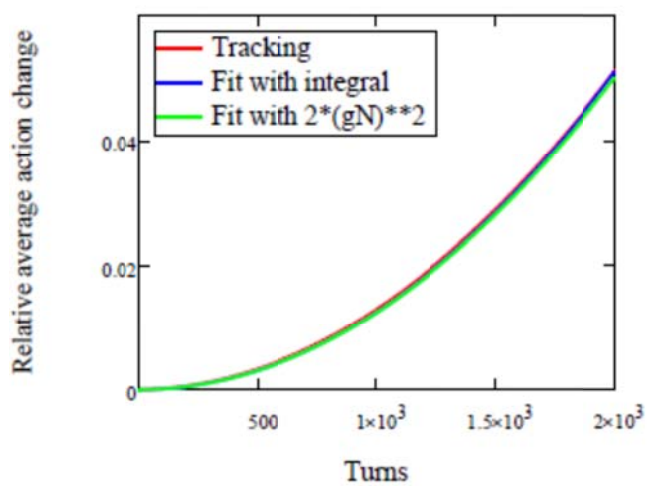

Figure 3: comparison of average action growth as calculated from eq. (38) and (39) with direct time domain tracking according to eq. (40) and (43).

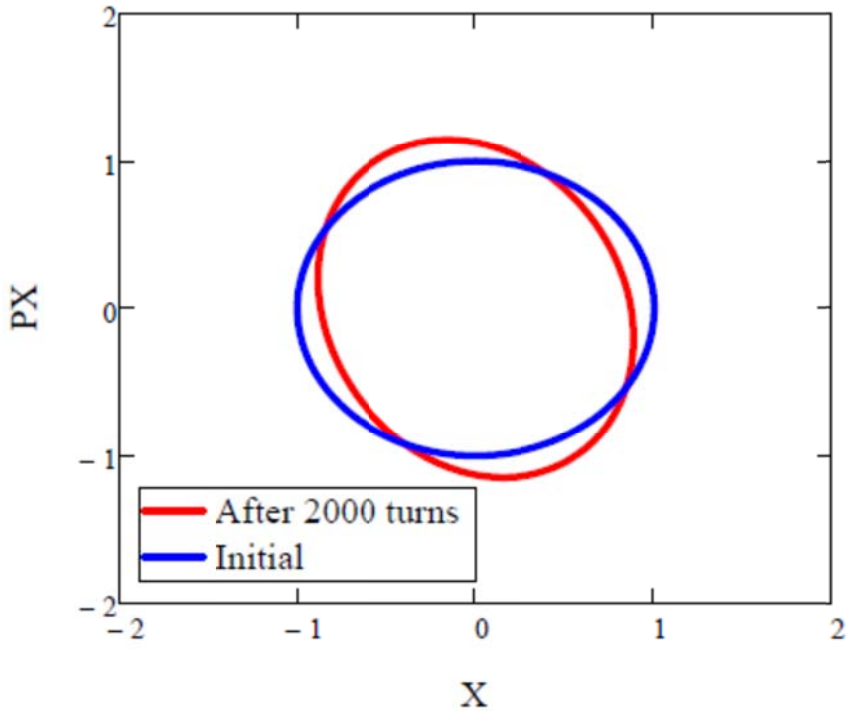

Figure 4: comparison of phase space distribution. The blue curve is the initial phase space distribution in normalized variables and the red curve is the phase space distribution after 2000 turns.

Requirement on Electron Bunch Repetition Frequency Accuracy

In order to store the ions for N_{store} turns with the action growth below a factor of two, the highest order resonances to avoid is given by eq. (39), i.e.

$$g_k = (2\pi)^{\frac{3}{2}} \frac{\sigma_e \Delta\nu_{peak}}{\Delta T_e} \mathrm{e}^{-\frac{2k^2\pi^2\sigma_e^2}{\Delta T_e^2}} < \frac{1}{\sqrt{2}N_{store}}, \tag{44}$$

which can be re-written as

$$k^2 > \frac{\Delta T_e^2}{2\pi^2\sigma_e^2} \ln\left[\frac{\sqrt{2}(2\pi)^{\frac{3}{2}} N_{store}\sigma_e \Delta\nu_{peak}}{\Delta T_e}\right]. \tag{45}$$

Eq. (45) suggests that the growth will always slower than the requirement if the space charge tune shift is smaller than

$$\Delta\nu_{peak} < \frac{\Delta T_e}{\sqrt{2}(2\pi)^{\frac{3}{2}} N_{store}\sigma_e}. \tag{46}$$

If eq. (46) is not satisfied, eq. (45) requires

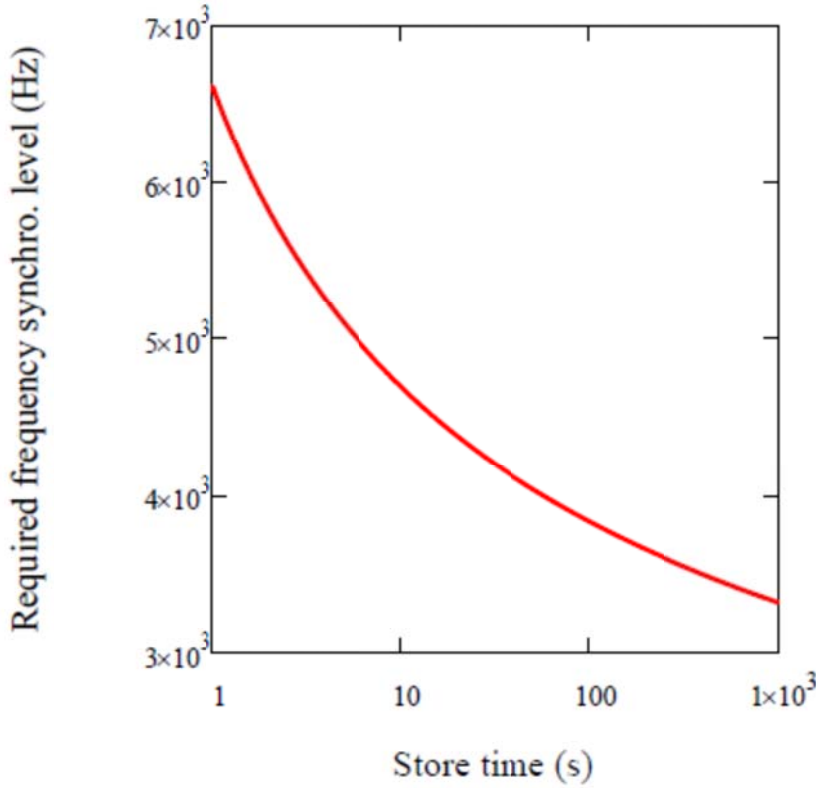

Figure 5: required electron repetition frequency accuracy level as a function of ion beam store time. The plot is generated from eq. (49) for $\Delta T_e = 10ns$, $\sigma_e = 1ns$, $Q_e = 4nC$, $\Delta\nu_{peak} = 3\times10^{-4}$ and $\gamma = 4.1$.

$$k > \frac{\Delta T_e}{\sqrt{2}\pi\sigma_e}\sqrt{\ln\left[\frac{\sqrt{2}\,(2\pi)^{\frac{3}{2}}\,N_{store}\sigma_e\Delta\nu_{peak}}{\Delta T_e}\right]}. \tag{47}$$

Assuming ν_e is close to zero, applying eq. (32) yields

$$\frac{\min\left(2\tilde{Q}_{dec},1-2\tilde{Q}_{dec}\right)}{\nu_e} > \frac{\Delta T_e}{\sqrt{2}\pi\sigma_e}\sqrt{\ln\left[\frac{\sqrt{2}\,(2\pi)^{\frac{3}{2}}\,N_{store}\sigma_e\Delta\nu_{peak}}{\Delta T_e}\right]}\ , \tag{48}$$

i.e.

$$\delta f_{rep} < \sqrt{2}\pi\cdot\frac{\sigma_e}{\Delta T_e}\cdot f_{rev}\cdot\min\left(2\tilde{Q}_{dec},1-2\tilde{Q}_{dec}\right)\cdot\left\{\ln\left[\frac{\sqrt{2}\,(2\pi)^{\frac{3}{2}}\,\sigma_e\Delta\nu_{peak}N_{store}}{\Delta T_e}\right]\right\}^{-\frac{1}{2}}. \tag{49}$$

Figure 5 shows the dependence of the required electron bunch repetition frequency as a function of RHIC ion beam storing time as calculated from eq. (49). For parameters listed in the caption of fig. 5, the required accuracy of the electron bunch repetition frequency is 3 KHz.

Estimation of Required Arriving Time Jitter Level

In the presence of arriving time jitter, eq. (13) becomes

$$\varepsilon_M = 4\pi \cdot \Delta\nu_{peak} \cdot \exp\left\{ -\frac{\left[\tau + M \cdot \nu_e \Delta T_e - Round\left(M \cdot \nu_e\right)\Delta T_e + \delta_{t,M}\right]^2}{2\sigma_e^2} \right\}, \tag{50}$$

with $\delta_{t,M}$ being the arriving time error in the Mth turn and τ is the location of the ion. If $\nu_e = 0$, eq. (50) becomes

$$\varepsilon_M = 4\pi \cdot \Delta\nu_{peak} \cdot \exp\left(-\frac{\left(\tau + \delta_{t,M}\right)^2}{2\sigma_e^2} \right). \tag{51}$$

The maximal kick variation happens at

$$\begin{aligned} \frac{d^2}{d\tau^2}\varepsilon_M &= 4\pi \cdot \Delta\nu_{peak} \cdot \frac{d}{d\tau}\left[-\frac{\tau}{\sigma_e^2}\exp\left(-\frac{\tau^2}{2\sigma_e^2} \right) \right] \\ &= 4\pi \cdot \Delta\nu_{peak} \cdot \frac{1}{\sigma_e^2}\exp\left(-\frac{\tau^2}{2\sigma_e^2} \right)\left[\frac{\tau^2}{\sigma_e^2} - 1 \right], \\ &= 0 \end{aligned} \tag{52}$$

i.e.

$$\tau_0 = \sigma_e . \tag{53}$$

If the arriving time jitter is much smaller than the bunch length, eq. (51) becomes

$$\begin{aligned} \varepsilon_M &= 4\pi \cdot \Delta\nu_{peak} \cdot \exp\left(-\frac{\left(1 + \delta_{t,M} / \sigma_e\right)^2}{2} \right) \\ &\approx 4\pi \cdot \Delta\nu_{peak} \cdot \exp\left(-\frac{\left(1 + 2\delta_{t,M} / \sigma_e\right)}{2} \right), \\ &= \frac{4\pi}{\sqrt{e}} \cdot \Delta\nu_{peak} \cdot \exp\left(-\frac{\delta_{t,M}}{\sigma_e} \right) \\ &\approx \frac{4\pi}{\sqrt{e}} \cdot \Delta\nu_{peak} \cdot \left(1 - \frac{\delta_{t,M}}{\sigma_e} \right) \end{aligned} \tag{54}$$

which leads to the average kick of

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i \approx \frac{4\pi}{\sqrt{e}} \cdot \Delta\nu_{peak} , \tag{55}$$

with the rms variation of

$$
\begin{aligned}
\left\langle \delta\varepsilon^2 \right\rangle &= \frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_i - \langle\varepsilon\rangle\right)^2 \\
&\approx \frac{1}{N}\sum_{i=1}^{N}\left(\frac{4\pi}{\sigma_e\sqrt{e}}\cdot\Delta\nu_{peak}\cdot\delta_{t,M}\right)^2 \\
&= \left(\frac{4\pi\Delta\nu_{peak}}{\sigma_e\sqrt{e}}\right)^2 \frac{1}{N}\sum_{i=1}^{N}\left(\delta_{t,M}\right)^2 \\
&= \left(\frac{4\pi\Delta\nu_{peak}}{\sigma_e\sqrt{e}}\right)^2 \cdot\left\langle \delta_t^2 \right\rangle
\end{aligned}
\tag{56}
$$

Eq. (13) of [1] reads

$$
\ln\left(\frac{J_N}{J_0}\right) = N\frac{\left\langle \delta\varepsilon^2 \right\rangle}{2}\frac{1-e^{-2\alpha}}{1+e^{-2\alpha}-2e^{-\alpha}\cos\left(4\pi Q\right)}, \tag{57}
$$

with α being the characteristic parameter of the time correlation of the jitter, i.e.

$$
\left\langle \delta\varepsilon_k \delta\varepsilon_{m+k} \right\rangle = \left\langle \delta\varepsilon^2 \right\rangle \exp\left(-\alpha|m|\right). \tag{58}
$$

Assuming $\alpha \to \infty$, i.e. there is no correlation between arriving time jitters for two different turns, eq. (57) reduces to

$$
\ln\left(\frac{J_N}{J_0}\right) = N\frac{\left\langle \delta\varepsilon^2 \right\rangle}{2}. \tag{59}
$$

Inserting eq. (56) into eq. (59) yields

$$
\begin{aligned}
J_N &= J_0 \exp\left\{\frac{N}{2}\left(\frac{4\pi\Delta\nu_{peak}}{\sigma_e\sqrt{e}}\right)^2 \cdot\left\langle \delta_t^2 \right\rangle\right\} \\
&\equiv J_0 \exp\left(\frac{t}{\tau_{grow}}\right)
\end{aligned}
, \tag{60}
$$

with the growing time of

$$
\tau_{grow} = \frac{e}{8\pi^2} T_{rev} \frac{\sigma_e^2}{\left\langle \delta_t^2 \right\rangle}\frac{1}{\Delta\nu_{peak}^2} . \tag{61}
$$

Requiring the growing time to be longer than the store time leads to

$$
\sqrt{\left\langle \delta_t^2 \right\rangle} < \frac{1}{2\pi}\frac{\sigma_e}{\Delta\nu_{peak}}\sqrt{\frac{e}{2N_{store}}} . \tag{62}
$$

Fig. 6 shows the required arriving time jitter level as a function of the beam store time. For the action growth below a factor of e, the required arrival time jitter is 37 ps.

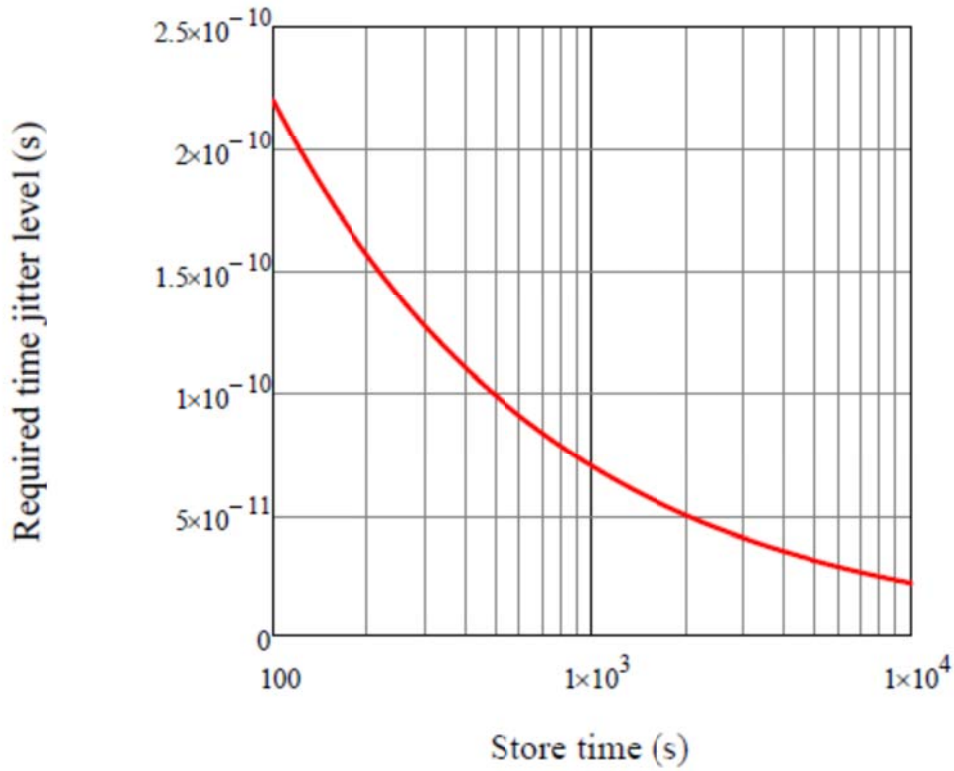

Figure 6: required time jitter level as calculated from eq. (62). It is assumed that there is no correlation between time jitters of two different turns.

Estimation of Required Peak Current Jitter Level

The maximal focusing strength is at the electron bunch center and is related to the electron bunch peak current by

$$\varepsilon_M = \alpha \cdot I_e(M,0), \tag{63}$$

with α defined in eq. (9). The rms variation of the focusing strength is thus

$$\left\langle \delta\varepsilon^2 \right\rangle = \frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_i - \langle\varepsilon\rangle\right)^2 = \alpha^2 \cdot \frac{1}{N}\sum_{i=1}^{N}\left(I_e(i,0) - \left\langle I_e(0)\right\rangle\right)^2 = \alpha^2 \left\langle \delta I_e(0)^2 \right\rangle . \tag{64}$$

According to eq. (14), the coefficient α can be written as

$$\alpha = 4\pi \frac{\left\langle \Delta\nu_{peak} \right\rangle}{\left\langle I_e(0) \right\rangle} . \tag{65}$$

Inserting eq. (65) into (64) produces

$$\left\langle \delta\varepsilon^2 \right\rangle = 16\pi^2 \left\langle \Delta\nu_{peak} \right\rangle^2 \frac{\left\langle \delta I_e(0)^2 \right\rangle}{\left\langle I_e(0) \right\rangle^2}, \tag{66}$$

which combines with eq. (57) leads to

$$\ln\left(\frac{J_N}{J_0}\right) = 8\pi^2 N \left\langle \Delta\nu_{peak} \right\rangle^2 \frac{\left\langle \delta I_e(0)^2 \right\rangle}{\left\langle I_e(0) \right\rangle^2} \frac{1-e^{-2\alpha}}{1+e^{-2\alpha}-2e^{-\alpha}\cos(4\pi Q)} \ . \tag{67}$$

Assuming peak current jitters from two different turns are uncorrelated, i.e. $\alpha \to \infty$, eq. (67) reduces to

$$\ln\left(\frac{J_N}{J_0}\right) = 8\pi^2 N \left\langle \Delta\nu_{peak} \right\rangle^2 \frac{\left\langle \delta I_e(0)^2 \right\rangle}{\left\langle I_e(0) \right\rangle^2} \ , \tag{68}$$

which can be rewritten as

$$J_N = J_0 \exp\left\{ 8\pi^2 N \left\langle \Delta\nu_{peak} \right\rangle^2 \frac{\left\langle \delta I_e(0)^2 \right\rangle}{\left\langle I_e(0) \right\rangle^2} \right\} \equiv J_0 \exp\left(\frac{N}{N_{grow}}\right) , \tag{69}$$

with

$$N_{grow} = \frac{1}{8\pi^2 \left\langle \Delta\nu_{peak} \right\rangle^2 \cdot \frac{\left\langle \delta I_e(0)^2 \right\rangle}{\left\langle I_e(0) \right\rangle^2}} \ . \tag{70}$$

Hence requiring the action growth time longer than the beam store time (or cooling time) gives the requirement for the current jitter as

$$\frac{\sqrt{\left\langle \delta I_e(0)^2 \right\rangle}}{\left\langle I_e(0) \right\rangle} < \frac{1}{2\pi \left\langle \Delta\nu_{peak} \right\rangle} \sqrt{\frac{T_{rev}}{2t_{store}}} \ . \tag{71}$$

Fig.7 plots the required peak current jitter level as calculated from eq. (71) with parameters given in the caption of fig. 5. As shown in fig. 7, in order to have the local growth time smaller than 1000 seconds, the required peak current jitter level is below 4.3%.

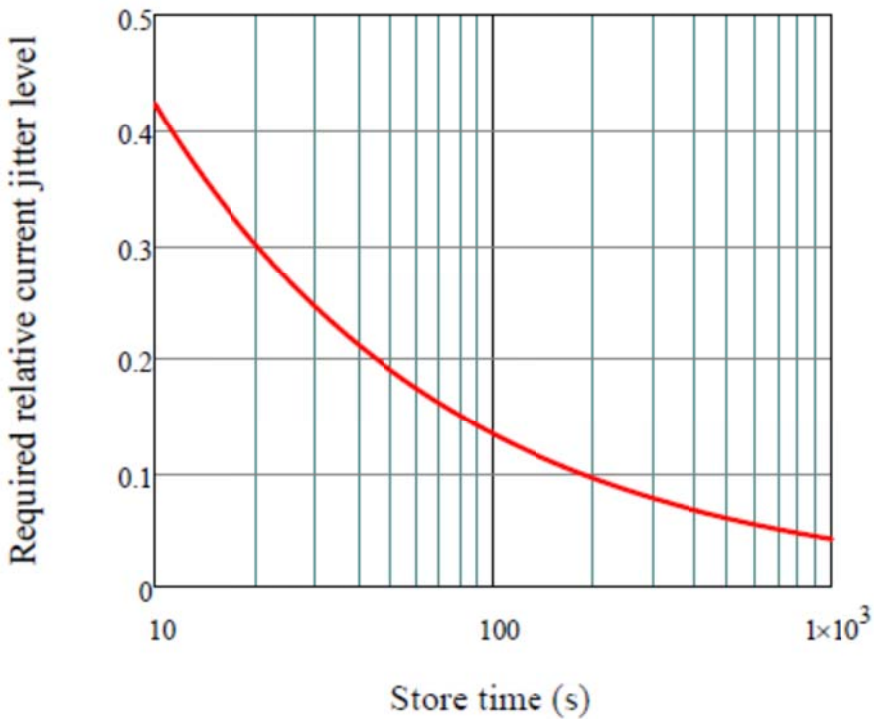

Figure 7: required relative peak current jitter level as calculated from eq. (71). The jitters from two different turns are assumed to be uncorrelated. The parameters applied are given in the caption of fig. 5.

APPENDIX A:

Now we will find the coefficient α so that $\delta\varepsilon_M$ has the same definition of reference [1]. According to eq. (1) of [1], the equation of motion reads

$$\frac{d^2}{d\theta^2}x = -Q^2 x - \delta_p(\theta) Q_x \varepsilon_M(\theta), \tag{72}$$

with

$$\theta = 2\pi t / T_{rev}. \tag{73}$$

Inserting eq. (73) into eq. (72) leads to

$$\begin{aligned}\frac{d^2}{dt^2}x &= -\frac{(2\pi)^2}{T_{rev}^2}Q^2 x - \frac{\delta_p(t)}{d\theta/dt}\frac{(2\pi)^2}{T_{rev}^2}Q\varepsilon_M(t)x \\ &= -\frac{(2\pi)^2}{T_{rev}^2}Q^2 x - \delta_p(t)\frac{(2\pi)}{T_{rev}}Q\varepsilon_M(t)x \\ &= -\omega_x^2 x - \delta_p(t)\omega_x\varepsilon_M(t)x\end{aligned} \quad . \tag{74}$$

Taking

$$x = A\cos\left[\left(Q+\Delta\nu_e\right)\omega_{rev}t\right] \tag{75}$$

and inserting it into eq. (74) leads to

$$\left(Q+\Delta\nu_e\left(t\right)\right)^2 = Q^2 + \delta_p\left(t\right)\frac{T_{rev}}{2\pi}Q\varepsilon_M\left(t\right) . \tag{76}$$

Integrating eq. (76) for the Mth turn leads to

$$\begin{aligned} &2\int_{(M-1)T_{rev}}^{MT_{rev}} \Delta\nu_e\left(t\right)dt = \frac{T_{rev}}{2\pi}\varepsilon_M\left(0\right) \\ &\Rightarrow \Delta\nu_{e,M} = \frac{1}{T_{rev}}\int_{(M-1)T_{rev}}^{MT_{rev}} \Delta\nu_e\left(t\right)dt = \frac{1}{4\pi}\varepsilon_M\left(0\right) \end{aligned} . \tag{77}$$

The space charge field from the electron beam in the co-moving frame is

$$\vec{E}_\perp' = \frac{en_e'}{2\varepsilon_0}\vec{r}_\perp' , \tag{78}$$

$$E_z' = 0 , \tag{79}$$

and

$$\vec{B}' = 0 . \tag{80}$$

Explicitly, eq. (78) can be written as

$$E_x' = \frac{-en_e'}{2\varepsilon_0}x' \tag{81}$$

and

$$E_y' = \frac{-en_e'}{2\varepsilon_0}y' . \tag{82}$$

In lab frame, the field is Lorentz transformed to

$$E_x = \frac{-en_e}{2\varepsilon_0}x , \tag{83}$$

$$E_y = \frac{-en_e}{2\varepsilon_0}y , \tag{84}$$

$$B_x = -\gamma\beta\frac{E_y'}{c} = \beta\frac{en_e}{2\varepsilon_0 c}y , \tag{85}$$

and

$$B_y = \gamma\beta\frac{E_x'}{c} = -\beta\frac{en_e}{2\varepsilon_0 c}x . \tag{86}$$

Thus the space charge force that the ions experienced from the electrons is

$$F_x = Z_i e\left(E_x - \beta c B_y\right) = -\frac{Z_i e^2 n_e}{2\varepsilon_0\gamma^2}x , \tag{87}$$

and

$$F_y = Z_i e\left(E_y + \beta c B_x\right) = -\frac{Z_i e^2 n_e}{2\varepsilon_0 \gamma^2} y \,. \tag{88}$$

The equation of motion is thus

$$\frac{d^2}{dt^2} x = -\omega_x^2 x + \frac{1}{m_i \gamma} F_x = -\omega_x^2 x - \frac{Z_i e^2 n_e L_{cool}}{2\varepsilon_0 m_i \gamma^3 \beta c} \delta_p\left(t\right) x \,. \tag{89}$$

Comparing eq. (89) with eq. (74) yields

$$\varepsilon_M\left(t\right) = \frac{Z_i e^2 n_e\left(t\right)}{2\varepsilon_0 m_i \gamma^3 \omega_x} = \frac{Z_i e^2 L_{cool} I_e\left(t\right)}{2\varepsilon_0 m_i \gamma^3 \omega_x \pi a_e^2 \left(\beta c\right)^2 e} \,. \tag{90}$$

Comparing eq. (90) with eq. (5) leads to

$$\alpha = \frac{Z_i e^2 L_{cool}}{2\varepsilon_0 m_i \gamma^3 \omega_x \pi a_e^2 \left(\beta c\right)^2} = \frac{2 r_0 \beta_{avg}}{\gamma^3 \beta^2 a_e^2} \frac{L_{cool}}{\beta c e} \,, \tag{91}$$

References

[1] M. Blaskiewicz, in C-A/AP/#363, 2009.